Exercise 4

The angular velocity $\mathbf{W}$ of a rotating solid body is a vector whose magnitude is the rate of angular displacement (radians per second) and whose direction is that in which a right-handed screw would advance if turned in the same direction. The position vector $\mathbf{r}$ of a point is the vector from the origin of coordinates to the point. Show that the velocity of any point in a rotating solid body is $\mathbf{v} = [\mathbf{W} \times \mathbf{r}]$, relative to an origin located on the axis of rotation.

Solution

Suppose we have a solid body rotating about an axis that points in the direction of the unit vector $\mathbf{n}$. Any point on this body will travel in a circle with respect to the axis of rotation. So if the body rotates through an angle of $\Delta\theta$, then the point's linear displacement will be the product of the circle's radius R and $\Delta\theta$.

$$\Delta s = R\Delta\theta$$

Divide both sides by Δt.

$$\frac{\Delta s}{\Delta t} = R\frac{\Delta\theta}{\Delta t}$$

Take the limit of both sides as $\Delta t \to 0$.

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} R\frac{\Delta\theta}{\Delta t}$$

Replace these limits with the corresponding derivatives.

$$\frac{ds}{dt} = R\frac{d\theta}{dt}$$

ds/dt is the rate that the linear displacement changes with respect to time, i.e. the speed $|\mathbf{v}|$.

$$|\mathbf{v}| = R\frac{d\theta}{dt}$$

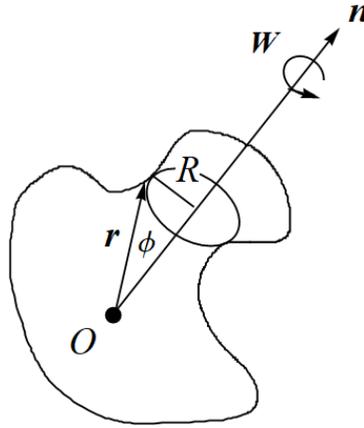The radius of the circle can be written in terms of the position vector, as can be seen from the figure below.

Figure 1: Schematic of the rotating solid body.

$$|\mathbf{v}| = |\mathbf{r}| \sin \phi \frac{d\theta}{dt},$$

where ϕ is the angle from the axis of rotation to the point on the body. Since the magnitude of the angular velocity is the rate that the angle changes with respect to time, that is,

$$|\mathbf{W}| = \frac{d\theta}{dt},$$

we have

$$|\mathbf{v}| = |\mathbf{r}| \sin \phi |\mathbf{W}|$$
$$= |\mathbf{r}||\mathbf{W}| \sin \phi.$$

This is the definition of the magnitude of the cross product.

$$|\mathbf{v}| = |\mathbf{r} \times \mathbf{W}| = |\mathbf{W} \times \mathbf{r}|$$

We have a choice of $\mathbf{r} \times \mathbf{W}$ or $\mathbf{W} \times \mathbf{r}$ for $\mathbf{v}$. We choose the latter expression because when we curl our fingers from $\mathbf{W}$ to $\mathbf{r}$ with our right hand, our thumb points in the direction that the point on the body is moving. Therefore,

$$\mathbf{v} = \mathbf{W} \times \mathbf{r}.$$